

Non-Fermi liquid behavior due to U(1) gauge field in two dimensions

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We study the damping rate of massless Dirac fermions due to the U(1) gauge field in (2+1)-dimensional quantum electrodynamics. In the absence of a Maxwell term for the gauge field, the fermion damping rate $\text{Im}\Sigma(\omega, T)$ is found to diverge in both perturbative and self-consistent results. In the presence of a Maxwell term, there is still divergence in the perturbative results for $\text{Im}\Sigma(\omega, T)$. Once the Maxwell term is included into the self-consistent equations for fermion self-energy and vacuum polarization functions, the fermion damping rate is free of divergence and exhibits non-Fermi liquid behavior: $\text{Im}\Sigma(\omega, T) \propto \max(\sqrt{\omega}, \sqrt{T})$.

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I. INTRODUCTION

In a normal Fermi liquid, the Landau quasiparticles are well-defined in the low-energy regime since their damping rate vanishes rapidly as $\text{Im}\Sigma(\omega, T) \propto \max(\omega^2, T^2)$ upon approaching the Fermi surface. Such rapidly diminishing damping rate is guaranteed by the Pauli exclusion principle and can be naturally understood by the fact that most of the states into which the fermions on Fermi surface would be scattered are already occupied by other fermions. Indeed, the states on the Fermi surface have an infinite lifetime and the low-lying quasiparticles can be safely considered as being nearly independent even when the interaction is not weak. Generally, one condition for the stability of Fermi liquid is the absence of singular or long-range interaction [1]. In normal metals, although the bare Coulomb interaction is long-ranged, it becomes short-ranged after including the dynamical screening effect. Therefore, the Fermi liquid theory can provide an excellent description for the electron liquid in metals.

Unlike Coulomb interaction, the long-range gauge interaction usually can not be fully screened in the absence of gauge symmetry breaking (it becomes short-ranged in a superconductor via Anderson-Higgs mechanism). It is thus possible that the unscreened gauge interaction would generate behaviors those are beyond the scope of Fermi liquid theory. In the context of ordinary metals, Holstein *et al.* first pointed out that non-Fermi liquid behavior arises from the coupling of electrons with the unscreened electromagnetic field [2]. This prominent result began the adventure for seeking various non-Fermi liquid behaviors in several different gauge theories [3–16] in the subsequent three decades.

The (2+1)-dimensional quantum electrodynamics (QED₃) of massless Dirac fermions is an interesting model that has been widely studied in both high energy physics [17, 18] and condensed matter physics [3–12]. The gauge field is strongly interacting with Dirac fermions, giving rise to rather unusual behaviors. As shown by Appelquist *et al.*, the massless Dirac fermions can acquire a finite mass via the mechanism of dynamical chiral symmetry breaking [17]. In the context of condensed matter physics, this phenomenon is usually identified as the formation of long-range antiferromagnetism in two-dimensional quantum Heisenberg antiferromagnet [6]. In the phase with chiral symmetry unbroken, the Dirac fermions are massless and the gauge field is strongly fluctuating but stays in the deconfined phase [18]. Now this field theory can describe the physics of a U(1) spin liquid, which is a novel state of matter without any spontaneous symmetry breaking. It is important to emphasize that the absence of symmetry breaking does not mean the absence of interesting physics. In fact, the U(1) spin liquid manifests a great deal of unusual physical properties and has been used to understand several correlated electron systems, including high temperature copper-oxide superconductors [3–9] and some spin-1/2 Kagome systems [12]. One of the most remarkable features of the gauge interaction is its ability to produce non-Fermi liquid behaviors. Owing to the chirality and linear spectrum of Dirac fermions, this interacting field theory is expected to display distinct behaviors compared with the much studied non-relativistic gauge systems [13–16].

In this paper, we study the possible non-Fermi liquid behavior by computing the damping rate $\text{Im}\Sigma(\omega, T)$ of massless Dirac fermion within the QED₃ theory. Generically, there are two kinds of QED₃ theories, depending on whether the theory has an explicit Maxwell term or not. First of all, as a well-defined quantum gauge field theory, it contains the Maxwell term $\propto F_{\mu\nu}F^{\mu\nu}$ explicitly in the action. If the theory is constructed by considering the quantum phase fluctuations in underdoped high temperature superconductors, then there is also an explicit Maxwell term in the Lagrangian [10, 11]. On the contrary, when the effective QED₃ theory is obtained by the slave-particle treatment of t - J model, there is no Maxwell term in the Lagrangian and the gauge field has its own dynamics only after integrating out the matter fields [3–7]. Here, we consider both of these two kinds QED₃ and show that they behave differently.

We first consider the QED₃ theory without Maxwell term. In the Coulomb gauge, the temporal and spatial components of U(1) gauge field are decoupled. We calculate the longitudinal and transverse fermion damping rates at both zero temperature, $T = 0$, and finite temperature. By straightforward perturbation computation, we find

that the damping rate is always divergent, either in the longitudinal contribution or in the transverse contribution. The appearance of divergences indicates the insufficiency of ordinary perturbative expansion in treating systems with singular interaction. Moreover, divergence still exists in the self-consistently coupled equations for fermion damping rate and gauge boson propagator.

We then consider the case with explicit Maxwell term for the gauge field. At the perturbative level, the fermion self-energy function diverges even in the presence of such term. After studying the self-consistent equations for fermion damping rate and gauge boson propagator, we found that the fermion damping rate is free of divergence and given by $\text{Im}\Sigma(\omega, T) \propto \max(\sqrt{\omega}, \sqrt{T})$, which is a non-Fermi liquid behavior.

We also discuss the damping rate of Dirac fermions due to the long-range Coulomb interaction. Although Coulomb interaction may be identified as the non-relativistic counterpart of the U(1) gauge interaction, they lead to different properties of Dirac fermions. Marginal Fermi liquid behavior is found by both perturbative and self-consistent approaches.

The damping rate of massless Dirac fermion is studied in QED₃ without Maxwell term in Sec. 2 and with Maxwell term in Sec. 3. In Sec. 4, we consider the Coulomb interaction and discuss why divergence appears in gauge theory, but not in such system. We end the paper with a summary and a brief discussion.

II. FERMION DAMPING RATE IN QED₃ WITHOUT MAXWELL TERM

The Lagrangian of QED₃ has the form

$$\mathcal{L} = \sum_{i=1}^N \Psi_i^\dagger (\partial_\tau - ia_0 - i\sigma \cdot (\partial - i\mathbf{a})) \Psi_i - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \quad (1)$$

In principle, the fermion field Ψ can be expressed in four-component or two-component representation. Since we only consider the chiral symmetric phase, we adopt the two-component representation of spinor field. The Dirac fermion flavor is taken to be a general N in order to perform $1/N$ expansion. In this paper, we discuss only non-compact QED₃ and hence there are no instantons. Note that the coupling between massless Dirac fermion and gauge field respects the Lorentz invariance, so there is no singular velocity renormalization [5]. When applied to high temperature superconductor, there is indeed an velocity anisotropy. However, this anisotropy turns out to be irrelevant, thus restoring the Lorentz invariance [19]. In this paper, the fermion velocities are simply taken to be unity.

As mentioned in the Introduction, we will first consider QED₃ theory without Maxwell term for gauge field. This is the effective low-energy theory of t - J model, obtained by using the slave-particle treatment. Now the $\propto F_{\mu\nu} F^{\mu\nu}$ term is simply dropped from the Lagrangian. The gauge field appearing in such model has its own dynamics only after integrating out the fermion fields, as well as other possible matter fields.

The Matsubara propagator of massless Dirac fermion is

$$G_0(\omega_n, \mathbf{k}) = \frac{1}{i\omega_n - \sigma \cdot \mathbf{k}}, \quad (2)$$

where $\omega_n = (2n+1)\pi T$ with n being integers. After analytic continuation, the retarded propagator reads

$$G_0(\omega, \mathbf{k}) = \frac{1}{\omega - \sigma \cdot \mathbf{k} + i\delta}. \quad (3)$$

For simplicity, the fermion energy is approximated by $\sigma \cdot \mathbf{k} \sim |\mathbf{k}|$. To decouple the temporal and spatial components of gauge field, it is convenient to work in the Coulomb gauge $k_i a_i = 0$. In the imaginary time formalism, the propagator for the gauge field can now be written as

$$D_{00}(\Omega_m, \mathbf{q}) = \frac{1}{D_1(\Omega_m, \mathbf{q})}, \quad (4)$$

$$D_{ij}(\Omega_m, \mathbf{q}) = \left(\delta_{ij} - \frac{q_i q_j}{\mathbf{q}^2} \right) \frac{1}{D_\perp(\Omega_m, \mathbf{q})} \quad (5)$$

where $\Omega_m = 2m\pi T$ for bosonic modes with m being integers. The vacuum polarization functions $D_1(\Omega_m, \mathbf{q})$ and $D_\perp(\Omega_m, \mathbf{q})$ come from the one-loop bubble diagram of Dirac fermions to the leading order of $1/N$ expansion. In particular, the polarization function appearing in the spatial component is given by

$$D_\perp(\Omega_m, \mathbf{q}) = D_2(\Omega_m, \mathbf{q}) - \frac{\Omega_m^2}{\mathbf{q}^2} D_1(\Omega_m, \mathbf{q}). \quad (6)$$

The functions $D_1(\Omega_m, \mathbf{q})$ and $D_2(\Omega_m, \mathbf{q})$ are defined as

$$D_1(\Omega_m, \mathbf{q}) = -NT \sum_{\omega_n} \int \frac{d^2 k}{(2\pi)^2} \text{Tr}[G_0(\omega_n, \mathbf{k}) G_0(\Omega_m + \omega_n, \mathbf{q} + \mathbf{k})], \quad (7)$$

$$D_2(\Omega_m, \mathbf{q}) = NT \sum_{\omega_n} \int \frac{d^2 k}{(2\pi)^2} \text{Tr}[\sigma_i G_0(\omega_n, \mathbf{k}) \sigma_i G_0(\Omega_m + \omega_n, \mathbf{q} + \mathbf{k})]. \quad (8)$$

At zero temperature, $T = 0$, it is straightforward to show that the retarded polarization functions have the forms

$$D_1(\Omega, \mathbf{q}) = \frac{N \mathbf{q}^2 \theta(|\mathbf{q}| - |\Omega|)}{16 \sqrt{|\mathbf{q}|^2 - \Omega^2}} + i \text{sgn} \Omega \frac{N \mathbf{q}^2 \theta(|\Omega| - |\mathbf{q}|)}{16 \sqrt{\Omega^2 - |\mathbf{q}|^2}}, \quad (9)$$

$$D_\perp(\Omega, \mathbf{q}) = \frac{N}{16} \theta(|\mathbf{q}| - |\Omega|) \sqrt{|\mathbf{q}|^2 - \Omega^2} - i \text{sgn} \Omega \frac{N}{16} \theta(|\Omega| - |\mathbf{q}|) \sqrt{\Omega^2 - |\mathbf{q}|^2}. \quad (10)$$

The fermion damping rate can be calculated by either the Fermi golden rule or the, basically equivalent but more formal, diagrammatic many-body technique [20]. We will utilize the latter one since it is easier to write down the self-consistent equations using diagrammatic technique.

A. Perturbative computation of fermion damping rate

We now calculate the fermion damping rate using conventional perturbative method. To the lowest order of $1/N$ expansion, the one-loop self-energy of Dirac fermion is given by Fig. 1 which can be written as

$$\Sigma(\omega_n, \mathbf{k}) = \Sigma_L(\omega_n, \mathbf{k}) + \Sigma_T(\omega_n, \mathbf{k}), \quad (11)$$

where

$$\Sigma_L(\omega_n, \mathbf{k}) = -T \sum_{\Omega_m} \int \frac{d^2 \mathbf{q}}{(2\pi)^2} G_0(\omega_n + \Omega_m, \mathbf{k} + \mathbf{q}) D_{00}(\Omega_m, \mathbf{q}) \quad (12)$$

$$\Sigma_T(\omega_n, \mathbf{k}) = T \sum_{\Omega_m} \int \frac{d^2 \mathbf{q}}{(2\pi)^2} \sigma_i G_0(\omega_n + \Omega_m, \mathbf{k} + \mathbf{q}) \sigma_j D_{ij}(\Omega_m, \mathbf{q}) \quad (13)$$

which represent the contribution from the longitudinal and transverse gauge field, respectively. The damping rate of massless Dirac fermion can be obtained by making analytic continuation, $i\omega_n \rightarrow \omega + i\delta$, as

$$\Sigma(\omega, \mathbf{k}) = \Sigma_L(\omega, \mathbf{k}) + \Sigma_T(\omega, \mathbf{k}), \quad (14)$$

and then taking the imaginary part, $\text{Im}\Sigma(\omega, \mathbf{k})$.

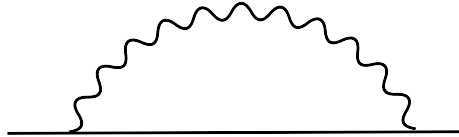


FIG. 1: Fermion self-energy correction to the leading order. The solid line represents the massless Dirac fermion, and the wiggly line represents the gauge field.

We first consider the transverse contribution to the damping rate at $T = 0$. Using the spectral representations for Dirac fermion and gauge boson propagators

$$G_0(\omega_n + \Omega_m, \mathbf{k} + \mathbf{q}) = - \int_{-\infty}^{+\infty} \frac{d\omega_1}{\pi} \frac{\text{Im} G_0(\omega_1, \mathbf{k} + \mathbf{q})}{i\omega_n + i\Omega_m - \omega_1}, \quad (15)$$

$$\frac{1}{D_\perp(\Omega_m, \mathbf{q})} = - \int_{-\infty}^{+\infty} \frac{d\omega_2}{\pi} \frac{\text{Im} \frac{1}{D_\perp(\omega_2, \mathbf{q})}}{i\Omega_m - \omega_2}, \quad (16)$$

the imaginary part of retarded self-energy function can be cast in the form

$$\begin{aligned} \text{Im}\Sigma_T(\omega, \mathbf{k}) = & \int \frac{d^2\mathbf{q}}{(2\pi)^2} \sigma_i \int_{-\infty}^{+\infty} \frac{d\omega_1}{\pi} \text{Im}[G_0(\omega_1, \mathbf{k} + \mathbf{q})] \sigma_j (\delta_{ij} - \frac{q_i q_j}{\mathbf{q}^2}) \text{Im} \left[\frac{1}{D_\perp(\omega_1 - \omega, \mathbf{q})} \right] \\ & \times [n_B(\omega_1 - \omega) + n_F(\omega_1)]. \end{aligned} \quad (17)$$

In the $T = 0$ limit, the occupation numbers simplify to

$$n_B(\omega_1 - \omega) + n_F(\omega_1) \rightarrow -\theta(\omega_1)\theta(\omega - \omega_1). \quad (18)$$

In the above expression, the imaginary retarded fermion propagator is $\text{Im}[G_0(\omega, \mathbf{k})] = -\pi\delta(\omega - |\mathbf{k}|)$, while the photon propagator has the form

$$\text{Im} \frac{1}{D_\perp(\Omega, \mathbf{q})} = \frac{16}{N \text{sgn}\Omega \sqrt{\Omega^2 - |\mathbf{q}|^2}} \theta(|\Omega| - |\mathbf{q}|). \quad (19)$$

After straightforward computation, the transverse fermion damping rate is found to be

$$\text{Im}\Sigma_T(\omega, \mathbf{k}) = -\frac{4}{N\pi^2} \int d^2\mathbf{q} \frac{1}{\sqrt{(\omega - |\mathbf{k} + \mathbf{q}|)^2 - |\mathbf{q}|^2}} \theta(\omega - |\mathbf{k} + \mathbf{q}| - |\mathbf{q}|). \quad (20)$$

To get an analytic expression, we first choose to use the frequently used on-shell approximation $\omega \equiv \omega_{\mathbf{k}} = |\mathbf{k}|$ and get

$$\text{Im}\Sigma_T(\omega_{\mathbf{k}}) = -\frac{4}{N\pi^2} \int d^2\mathbf{q} \frac{1}{\sqrt{(|\mathbf{k}| - |\mathbf{k} + \mathbf{q}|)^2 - |\mathbf{q}|^2}} \theta(|\mathbf{k}| - |\mathbf{k} + \mathbf{q}| - |\mathbf{q}|). \quad (21)$$

Since the step function always satisfies $\theta(|\mathbf{k}| - |\mathbf{k} + \mathbf{q}| - |\mathbf{q}|) \equiv 0$, we know that $\text{Im}\Sigma_T(\omega_{\mathbf{k}}) \equiv 0$. More generally, $\theta(\omega - |\mathbf{k} + \mathbf{q}| - |\mathbf{q}|) \equiv 0$ for $|\omega| \leq |\mathbf{k}|$, so that

$$\text{Im}\Sigma_T(\omega \leq |\mathbf{k}|, \mathbf{k}) \equiv 0. \quad (22)$$

In order to get a finite analytic expression, we use the zero-momentum limit $|\mathbf{k}| = 0$ and finally have

$$\text{Im}\Sigma_T(\omega) = -\frac{8}{N\pi} \int_0^{\frac{\omega}{2}} d|\mathbf{q}| \frac{|\mathbf{q}|}{\sqrt{\omega^2 - 2\omega|\mathbf{q}|}} = -\frac{8\omega}{3N\pi}. \quad (23)$$

This linear-in-energy expression clearly signals a marginal Fermi liquid behavior [21].

This result is obtained for $\omega > 0$. When $\omega < 0$, the expression should be $\text{Im}\Sigma_T(\omega) = \frac{8}{3N\pi}\omega$. In general, the fermion damping rate has the form $\text{Im}\Sigma_T(\omega) = -\frac{8}{3N\pi}|\omega|$. Without loss of generality, we consider only positive ω in the following. The real part of retarded fermion self-energy can be directly obtained by the Kramers-Kronig relation, as $\text{Re}\Sigma_T(\omega) \propto \omega \ln \omega$.

This contribution has its own physical application. In the effective gauge theory derived by the slave-boson approach, the gauge field also couples to non-relativistic scalar bosons which describe the motion of charged holons [3, 5–7]. The scalar bosons are incompressible and thus effectively screen the temporal component of the gauge field. So the temporal component a_0 can be omitted, but the transverse components \mathbf{a} remain unscreened and should be carefully treated. Within this effective field theory, several interesting results have been obtained, including the singular corrections to specific heat and susceptibility [5], and the algebraic correlation [7]. The above results show that the transverse gauge field leads to marginal Fermi liquid behavior at $T = 0$.

It seems impossible to get an analytical expression for the general damping rate $\text{Im}\Sigma_T(\omega, \mathbf{k})$, so we compute it by numerical skills with the results being shown in Fig. 2.

The longitudinal contribution to fermion self-energy function will be calculated analogously. The longitudinal damping rate can be written as

$$\text{Im}\Sigma_L(\omega, \mathbf{k}) = \int \frac{d^2\mathbf{q}}{(2\pi)^2} \int_0^\omega \frac{d\omega_1}{\pi} \text{Im}[G_0(\omega_1, \mathbf{k} + \mathbf{q})] \text{Im} \left[\frac{1}{D_1(\omega_1 - \omega, \mathbf{q})} \right]. \quad (24)$$

Using the expression for temporal gauge propagator

$$\text{Im} \left[\frac{1}{D_1(\Omega, \mathbf{q})} \right] = \frac{-16\sqrt{\Omega^2 - |\mathbf{q}|^2}}{\text{sgn}\Omega N |\mathbf{q}|^2} \theta(|\Omega| - |\mathbf{q}|), \quad (25)$$

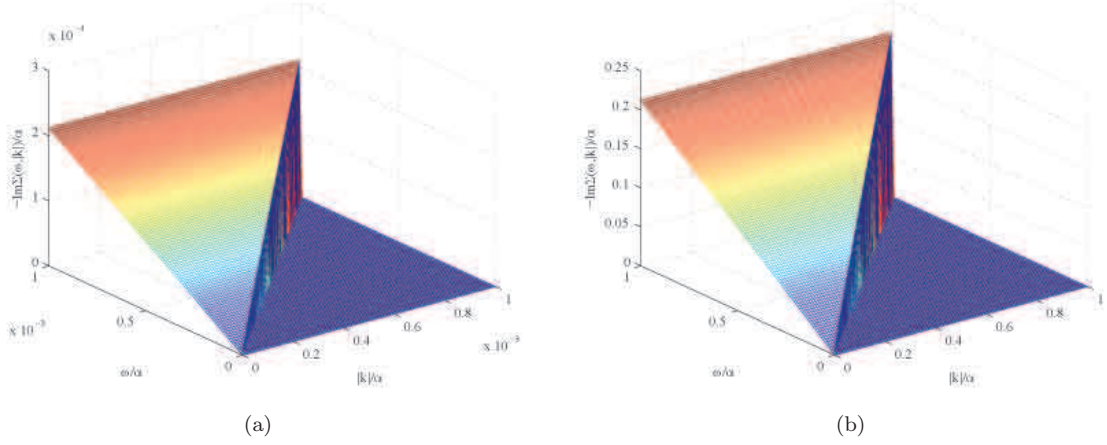


FIG. 2: Transverse contribution to damping rate of Dirac fermion without Maxwell term of gauge field at $T = 0$.

we obtain

$$\text{Im}\Sigma_L(\omega, \mathbf{k}) = -\frac{4}{N\pi^2} \int d^2\mathbf{q} \frac{\sqrt{(\omega - |\mathbf{k} + \mathbf{q}|)^2 - |\mathbf{q}|^2}}{|\mathbf{q}|^2} \theta(\omega - |\mathbf{k} + \mathbf{q}| - |\mathbf{q}|). \quad (26)$$

Analogous to the transverse contribution, we can easily get

$$\text{Im}\Sigma_L(\omega \leq |\mathbf{k}|, \mathbf{k}) \equiv 0. \quad (27)$$

In the limit $|\mathbf{k}| = 0$, the longitudinal damping rate is

$$\text{Im}\Sigma_L(\omega) = -\frac{8}{N\pi} \int_0^{\frac{\omega}{2}} d|\mathbf{q}| \frac{\sqrt{\omega^2 - 2\omega|\mathbf{q}|}}{|\mathbf{q}|}. \quad (28)$$

Obviously there appears a serious infrared divergence. In fact, $\text{Im}\Sigma_L(\omega, \mathbf{k})$ is always infrared divergent for $\omega > |\mathbf{k}|$. The general, $\text{Im}\Sigma_L(\omega, \mathbf{k})$ can be written as

$$\text{Im}\Sigma_L(\omega, \mathbf{k}) = -\frac{4}{N\pi^2} \int_0^{+\infty} d|\mathbf{q}| F_1(|\mathbf{q}|) \quad (29)$$

with

$$F_1(|\mathbf{q}|) = \int_0^{2\pi} d\varphi \frac{\sqrt{(\omega - |\mathbf{k} + \mathbf{q}|)^2 - |\mathbf{q}|^2}}{|\mathbf{q}|} \theta(\omega - |\mathbf{k} + \mathbf{q}| - |\mathbf{q}|). \quad (30)$$

Where φ is the angle between \mathbf{q} and \mathbf{k} . When $\omega > |\mathbf{k}|$,

$$\lim_{|\mathbf{q}| \rightarrow 0} F_1(|\mathbf{q}|) = \frac{2\pi(\omega - |\mathbf{k}|)}{|\mathbf{q}|}, \quad (31)$$

which is divergent in the infrared region. The general damping rate $\text{Im}\Sigma_L(\omega, \mathbf{k})$ are shown in Fig. 3.

Here, an infrared divergence appears in the expression of fermion damping rate due to the singular gauge interaction. Such divergent damping rate is surely not well-defined at $T = 0$. This does not imply that the damping rate itself diverges, but reflects the inefficiency of naive perturbation computation.

We now extend the above consideration to finite temperature, $T \gg \omega$. We first calculate the longitudinal damping rate of Dirac fermions. After a series of manipulations, in the limit $|\mathbf{k}| = 0$, the longitudinal component of imaginary self-energy function is written in the form

$$\text{Im}\Sigma_L(\omega, T) = \frac{1}{2\pi} \int_0^{+\infty} d|\mathbf{q}| |\mathbf{q}| \text{Im} \left[\frac{1}{D_1(|\mathbf{q}| - \omega, \mathbf{q}, T)} \right] [n_B(|\mathbf{q}| - \omega) + n_F(|\mathbf{q}|)]. \quad (32)$$

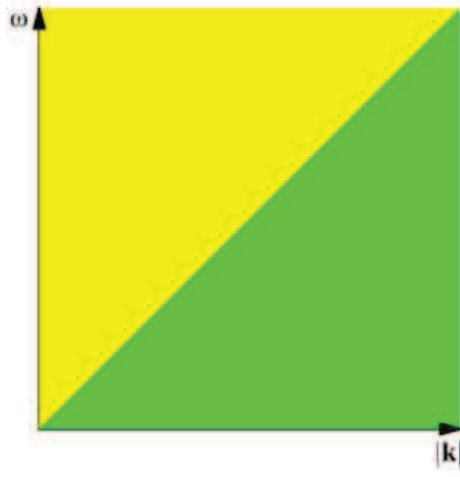


FIG. 3: In the yellow region where $\omega > |\mathbf{k}|$, $\text{Im}\Sigma_L(\omega, \mathbf{k})$ is infrared divergent. In the green region where $\omega \leq |\mathbf{k}|$, $\text{Im}\Sigma_L(\omega, \mathbf{k}) = 0$.

Notice that the occupation numbers $n_B(|\mathbf{q}| - \omega)$ and $n_F(|\mathbf{q}|)$ damp exponentially with $\frac{|\mathbf{q}|}{T}$, so the dominant contribution of the integral comes from the domain $|\mathbf{q}| < T$. Hence the ultraviolet cutoff of the integral can be set to be T . In the high temperature limit, it is convenient to make the simplifications

$$\begin{aligned} n_B(|\mathbf{q}| - \omega) &\approx \frac{T}{|\mathbf{q}| - \omega}, \\ n_F(|\mathbf{q}|) &\approx \frac{1}{2}. \end{aligned} \quad (33)$$

The polarization function $D_1(|\mathbf{q}| - \omega, |\mathbf{q}|, T)$ in the limit $T \gg \omega$ has a very complicated expression, which makes analytical computation difficult. To keep the analytic tractability, it is necessary to make some approximations. To simplify the analysis, we divide the whole domain of $|\mathbf{q}|$ into three sections and then perform the momentum integration respectively. Specifically, we decompose the longitudinal damping rate into the following three parts:

$$\begin{aligned} \text{Im}\Sigma_L(\omega, T) &= \frac{1}{2\pi} \left(\int_0^{\frac{\omega}{2}} + \int_{\frac{\omega}{2}}^{q^*} + \int_{q^*}^T \right) d|\mathbf{q}| |\mathbf{q}| \text{Im} \left[\frac{1}{D_1(|\mathbf{q}| - \omega, \mathbf{q}, T)} \right] [n_B(|\mathbf{q}| - \omega) + n_F(|\mathbf{q}|)] \\ &= I_1 + I_2 + I_3. \end{aligned} \quad (34)$$

Here, the variable q^* is a particularly chosen quantity between ω and T which we employ to simply the calculation. The key motivation to introduce this quantity is to specify the most important contribution of the integration. In principle, we can employ any q^* between ω and T . In this paper, we assume that q^* satisfy

$$\sqrt{\frac{T}{q^*}} \gg \frac{q^*}{\omega} \gg 1. \quad (35)$$

The polarization functions are given by the following expressions. When $0 < |\mathbf{q}| < \frac{\omega}{2}$,

$$\text{Re}D_1(|\mathbf{q}| - \omega, |\mathbf{q}|, T) \approx C_1 T - C_2 \frac{T(\omega - |\mathbf{q}|)}{\sqrt{\omega^2 - 2\omega|\mathbf{q}|}} \quad (36)$$

$$\text{Im}D_1(|\mathbf{q}| - \omega, |\mathbf{q}|, T) \approx -C_3 \frac{|\mathbf{q}|^2(\omega - |\mathbf{q}|)}{T\sqrt{\omega^2 - 2\omega|\mathbf{q}|}} \quad (37)$$

When $\frac{\omega}{2} < |\mathbf{q}| < q^*$,

$$\text{Re}D_1(|\mathbf{q}| - \omega, |\mathbf{q}|, T) \approx C_1 T + C_4 \frac{|\mathbf{q}|^3}{T\sqrt{2\omega|\mathbf{q}| - \omega^2}} \quad (38)$$

$$\text{Im}D_1(|\mathbf{q}| - \omega, |\mathbf{q}|, T) \approx C_2 \frac{T(|\mathbf{q}| - \omega)}{\sqrt{2\omega|\mathbf{q}| - \omega^2}} \quad (39)$$

When $q^* < |\mathbf{q}| < T$,

$$\text{Re}D_1(|\mathbf{q}| - \omega, |\mathbf{q}|, T) \approx C_1 T + C_3 \frac{|\mathbf{q}|^{\frac{5}{2}}}{T\sqrt{2\omega}} \quad (40)$$

$$\text{Im}D_1(|\mathbf{q}| - \omega, |\mathbf{q}|, T) \approx C_2 \frac{T\sqrt{|\mathbf{q}|}}{\sqrt{2\omega}} \quad (41)$$

where $C_1 = N \frac{\ln 2}{\pi}$, $C_2 = \frac{N}{\pi} \left(\frac{1}{8} + 2e^{-1} \right)$, $C_3 = \frac{N}{64}$, $C_4 = \frac{N}{48\pi}$. After direct calculations, we find

$$I_1 \propto \frac{\omega^3}{T^2}, \quad (42)$$

$$I_2 \propto \frac{q^{*\frac{3}{2}}}{\sqrt{\omega}}, \quad (43)$$

$$I_3 \propto \sqrt{\omega T}. \quad (44)$$

It is easy to check that $|I_3| \gg |I_2| \gg |I_1|$. Therefore, the temporal damping rate at $T \gg \omega$ has the form

$$\text{Im}\Sigma_L(\omega, T) \propto \sqrt{\omega T}. \quad (45)$$

This expression is surely not of the normal Fermi liquid type. It is, however, not the standard marginal Fermi liquid behavior.

We next consider the transverse component of fermion damping rate. Similar to treatments presented above, the momentum $|\mathbf{q}|$ should also be divided into three sections, so that in the limit $|\mathbf{k}| = 0$, the damping rate $\text{Im}\Sigma_T(\omega, T)$ has the form

$$\begin{aligned} \text{Im}\Sigma_T(\omega, T) &= -\frac{1}{2\pi} \left(\int_0^{\frac{\omega}{2}} + \int_{\frac{\omega}{2}}^{q^*} + \int_{q^*}^T \right) d|\mathbf{q}| |\mathbf{q}| \text{Im} \left[\frac{1}{D_\perp(|\mathbf{q}| - \omega, \mathbf{q}, T)} \right] [n_B(|\mathbf{q}| - \omega) + n_F(|\mathbf{q}|)] \\ &= I'_1 + I'_2 + I'_3. \end{aligned} \quad (46)$$

The approximate expression for $D_\perp(|\mathbf{q}| - \omega, \mathbf{q}, T)$ were shown as follow. When $0 < |\mathbf{q}| < \frac{\omega}{2}$,

$$\text{Re}D_\perp(|\mathbf{q}| - \omega, |\mathbf{q}|, T) \approx C_1 \frac{T(\omega - |\mathbf{q}|)^2}{|\mathbf{q}|^2} - C_2 \frac{T(\omega - |\mathbf{q}|)\sqrt{\omega^2 - 2\omega|\mathbf{q}|}}{|\mathbf{q}|^2} \quad (47)$$

$$\text{Im}D_\perp(|\mathbf{q}| - \omega, |\mathbf{q}|, T) \approx C_3 \frac{(\omega - |\mathbf{q}|)\sqrt{\omega^2 - 2\omega|\mathbf{q}|}}{T} \quad (48)$$

When $\frac{\omega}{2} < |\mathbf{q}| < q^*$,

$$\text{Re}D_\perp(|\mathbf{q}| - \omega, |\mathbf{q}|, T) \approx C_1 \frac{T(|\mathbf{q}| - \omega)^2}{|\mathbf{q}|^2} + C'_4 \frac{|\mathbf{q}|\sqrt{2\omega|\mathbf{q}| - \omega^2}}{T} \quad (49)$$

$$\text{Im}D_\perp(|\mathbf{q}| - \omega, |\mathbf{q}|, T) \approx -C_2 \frac{T(|\mathbf{q}| - \omega)\sqrt{2\omega|\mathbf{q}| - \omega^2}}{|\mathbf{q}|^2} \quad (50)$$

When $q^* < |\mathbf{q}| < T$,

$$\text{Re}D_\perp(|\mathbf{q}| - \omega, |\mathbf{q}|, T) \approx C_1 T \quad (51)$$

$$\text{Im}D_\perp(|\mathbf{q}| - \omega, |\mathbf{q}|, T) \approx -C_2 \frac{T\sqrt{2\omega}}{\sqrt{|\mathbf{q}|}} \quad (52)$$

where $C'_4 = \frac{N}{24\pi}$. After direct calculation, we can get

$$I'_1 \propto \frac{\omega^3}{T^2} \quad (53)$$

$$I'_2 \propto \frac{T^2}{\omega} \quad (54)$$

$$I'_3 \propto \sqrt{\omega T} \quad (55)$$

It is clear that $|I'_2| \gg |I'_3| \gg |I'_1|$, then we can conclude that, in the limit $T \gg \omega$

$$\text{Im}\Sigma_T(\omega, T) \propto \frac{T^2}{\omega}. \quad (56)$$

This expression is also divergent at zero energy $\omega \rightarrow 0$.

The above calculations were carried out in the zero-momentum limit $|\mathbf{k}| = 0$. Using the on-shell approximation $\omega \equiv \omega_{\mathbf{k}} = |\mathbf{k}|$ at finite temperature $T \gg \omega_{\mathbf{k}}$, we find that $\text{Im}\Sigma_L(\omega_{\mathbf{k}}, T) \propto \sqrt{\omega_{\mathbf{k}}T}$, but that $\text{Im}\Sigma_T(\omega_{\mathbf{k}}, T)$ is divergent in the infrared region. If we instead work in the zero-energy limit $\omega = 0$ at $T \gg |\mathbf{k}|$, then we find that $\text{Im}\Sigma_L(|\mathbf{k}|, T) \propto \sqrt{|\mathbf{k}|T}$ and $\text{Im}\Sigma_T(|\mathbf{k}|, T) \propto \sqrt{|\mathbf{k}|T}$.

B. Self-consistent computation of fermion damping rate

The perturbative results for Dirac fermion damping rate always contain divergence, at both zero and finite temperature. It seems difficult to eliminate these divergences by including higher order corrections because they arise essentially from the singular gauge interaction. It is worth pointing out that such divergence exists in a wide range of physical problems. In the interacting electron gas, the electron self-energy diverges at zero energy if the bare, long-range Coulomb interaction is considered. In realistic metals, however, the bare Coulomb potential is always replaced by a short-ranged Yukawa potential after including the dynamical screening effect. The Debye screening ensures the infrared safety of the problem. In the present issue, however, the gauge interaction remains long-ranged even after taking the dynamical screening into account. Indeed, the gauge invariance ensures the masslessness of the U(1) gauge boson. Analogous divergence also appears in the perturbative computation for the damping rate of non-relativistic spinons due to scattering by gauge field [15].

There is also an infrared divergence when computing perturbatively the self-energy of Dirac fermion in disordered potential [22]. Such divergence is essentially due to the linear spectrum of Dirac fermions. The most popular method to overcome this divergence is to invoke the so-called self-consistent Born approximation, which can give rise to a finite scattering rate [22]. It is natural to ask whether similar self-consistent approach can be used to eliminate the infrared divergence appearing in the present problem. To answer this question, we replace the internal free fermion propagator of Fig.1 by the following retarded propagator

$$G(\omega, \mathbf{k}) = \frac{1}{\omega - \sigma \cdot \mathbf{k} - i\text{Im}\Sigma(\omega, T)}, \quad (57)$$

and then construct an integral equation for the damping rate $\text{Im}\Sigma(\omega, T)$. If the perturbative expressions for the polarization functions are used, then only trivial result can be obtained after numerical computation. This is easy to understand by noting the fact that the infrared divergence originates from the singular gauge interaction.

All the above computations are based on the polarization functions obtained from the free propagator of Dirac fermion. The feedback of fermion damping due to gauge field is completely neglected when calculating the dynamically screened gauge field and fermion self-energy function. However, if the fermion damping is really significant, as it should be in a non-Fermi liquid, its effect can not be simply neglected. It is conceivable to speculate that the divergences appearing in the Dirac fermion self-energy might be eliminated by incorporating the feedback effect of fermion damping. We now turn to such kind of self-consistent treatment of Dirac fermions and gauge bosons. Formally, the fermion damping rate satisfies the integral equation

$$\begin{aligned} \text{Im}\Sigma(\omega, T) = & \int \frac{d^2\mathbf{q}}{(2\pi)^2} \int_{-\infty}^{+\infty} \frac{d\omega_1}{\pi} \frac{\text{Im}\Sigma(\omega_1, T)}{(\omega_1 - |\mathbf{q}|)^2 + (\text{Im}\Sigma(\omega_1, T))^2} \\ & \times \text{Im} \left[-\frac{1}{D_1(\omega_1 - \omega, \mathbf{q}, T)} + \frac{1}{D_{\perp}(\omega_1 - \omega, \mathbf{q}, T)} \right] [n_B(\omega_1 - \omega) + n_F(\omega_1)]. \end{aligned} \quad (58)$$

Using the fermion propagator (57), the polarization functions $\text{Im}D_1$ and $\text{Im}D_{\perp}$ can be constructed as follows

$$\begin{aligned} & \text{Im}D_1(\varepsilon, |\mathbf{q}|, T) \\ = & 2N \int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{d\omega_1}{\pi} \left\{ \text{Im} \left[\frac{\omega_1 - i\text{Im}\Sigma(\omega_1, T)}{(\omega_1 - i\text{Im}\Sigma(\omega_1, T))^2 - |\mathbf{k}|^2} \right] \text{Im} \left[\frac{\omega_1 + \varepsilon - i\text{Im}\Sigma(\omega_1 + \varepsilon, T)}{(\omega_1 + \varepsilon - i\text{Im}\Sigma(\omega_1 + \varepsilon, T))^2 - |\mathbf{k} + \mathbf{q}|^2} \right] \right. \\ & \left. + \mathbf{k} \cdot (\mathbf{k} + \mathbf{q}) \text{Im} \left[\frac{1}{(\omega_1 - i\text{Im}\Sigma(\omega_1, T))^2 - |\mathbf{k}|^2} \right] \text{Im} \left[\frac{1}{(\omega_1 + \varepsilon - i\text{Im}\Sigma(\omega_1 + \varepsilon, T))^2 - |\mathbf{k} + \mathbf{q}|^2} \right] \right\} \end{aligned}$$

$$\times [n_F(\omega_1) - n_F(\omega_1 + \varepsilon)]. \quad (59)$$

$$\begin{aligned} & \text{Re} D_1(\varepsilon, |\mathbf{q}|, T) \\ = & 2N \int \frac{d^2 \mathbf{k}}{(2\pi)^2} \frac{d\omega_1}{\pi} \left\{ \text{Im} \left[\frac{\omega_1 - i\text{Im}\Sigma(\omega_1, T)}{(\omega_1 - i\text{Im}\Sigma(\omega_1, T))^2 - |\mathbf{k}|^2} \right] \text{Re} \left[\frac{\omega_1 + \varepsilon - i\text{Im}\Sigma(\omega_1 + \varepsilon, T)}{(\omega_1 + \varepsilon - i\text{Im}\Sigma(\omega_1 + \varepsilon, T))^2 - |\mathbf{k} + \mathbf{q}|^2} \right] \right. \\ & \left. + \mathbf{k} \cdot (\mathbf{k} + \mathbf{q}) \text{Im} \left[\frac{1}{(\omega_1 - i\text{Im}\Sigma^{\text{ret}}(\omega_1, T))^2 - |\mathbf{k}|^2} \right] \text{Re} \left[\frac{1}{(\omega_1 + \varepsilon - i\text{Im}\Sigma^{\text{ret}}(\omega_1 + \varepsilon, T))^2 - |\mathbf{k} + \mathbf{q}|^2} \right] \right\} \\ & \times [n_F(\omega_1) - n_F(\omega_1 + \varepsilon)], \end{aligned} \quad (60)$$

$$\begin{aligned} & \text{Im} D_2(\varepsilon, |\mathbf{q}|, T) \\ = & -4N \int \frac{d^2 \mathbf{k}}{(2\pi)^2} \frac{d\omega_1}{\pi} \text{Im} \left[\frac{\omega_1 - i\text{Im}\Sigma(\omega_1, T)}{(\omega_1 - i\text{Im}\Sigma(\omega_1, T))^2 - |\mathbf{k}|^2} \right] \text{Im} \left[\frac{\omega_1 + \varepsilon - i\text{Im}\Sigma(\omega_1 + \varepsilon, T)}{(\omega_1 + \varepsilon - i\text{Im}\Sigma(\omega_1 + \varepsilon, T))^2 - |\mathbf{k} + \mathbf{q}|^2} \right] \\ & \times [n_F(\omega_1) - n_F(\omega_1 + \varepsilon)], \end{aligned} \quad (61)$$

$$\begin{aligned} & \text{Re} D_2(\varepsilon, |\mathbf{q}|, T) \\ = & -4N \int \frac{d^2 \mathbf{k}}{(2\pi)^2} \frac{d\omega_1}{\pi} \text{Im} \left[\frac{\omega_1 - i\text{Im}\Sigma(\omega_1, T)}{(\omega_1 - i\text{Im}\Sigma(\omega_1, T))^2 - |\mathbf{k}|^2} \right] \text{Re} \left[\frac{\omega_1 + \varepsilon - i\text{Im}\Sigma(\omega_1 + \varepsilon, T)}{(\omega_1 + \varepsilon - i\text{Im}\Sigma(\omega_1 + \varepsilon, T))^2 - |\mathbf{k} + \mathbf{q}|^2} \right] \\ & \times [n_F(\omega_1) - n_F(\omega_1 + \varepsilon)], \end{aligned} \quad (62)$$

where

$$D_{\perp}(\varepsilon, |\mathbf{q}|, T) = D_2(\varepsilon, |\mathbf{q}|, T) + \frac{\varepsilon^2}{|\mathbf{q}|^2} D_1(\varepsilon, |\mathbf{q}|, T). \quad (63)$$

These equations appear to be very complicated and hard to solve. To get the fermion damping rate from these coupled integral equations, we find it is convenient to employ a simple dimensional analysis similar to that used by Vojta *et al.* [23]. After dividing all momenta, energy, and self-energy function by ω at $T = 0$ and by T at finite T , it is found that $\text{Im}\Sigma(\omega) \propto \omega$ and $\text{Im}\Sigma(T) \propto T$ respectively. Intuitively, this re-scaling procedure reflects the typical behavior of a marginal Fermi liquid. However, if we define

$$\frac{\text{Im}\Sigma(\omega)}{\omega} = A, \quad (64)$$

$$\frac{\text{Im}\Sigma(T)}{T} = B, \quad (65)$$

then numerical calculation find no convergent solutions for A and B . It turns out that, although formally the damping rate depends linearly on ω at $T = 0$ and on T at $\omega = 0$, a divergence appears in the regime where the energy of polarization functions vanishes. Specifically, this divergence emerges when $\omega_1 \rightarrow \omega$ in the fermion damping rate equation (58). Since the functions D_1 and D_{\perp} appear in equation (58) as denominators, the fermion damping rate actually diverges as the energy of polarization functions vanishes. This qualitative analysis is confirmed by the numerical computations.

In the above, we show that it is hard to get meaningful results of fermion damping rate within both perturbation theory and self-consistent treatment when the gauge field has no explicit Maxwell term. There is always some kind of divergence. All these divergences originate from the dynamically screened propagator of gauge field. The fermion damping rate seems not to be a well-defined quantity, at least under the approximations considered in the above.

III. FERMION DAMPING RATE IN QED₃ WITH MAXWELL TERM

In this section, we study the QED₃ theory with explicit Maxwell term. According to the standard framework of relativistic quantum field theory, it is natural to keep such kinetic term in the Lagrangian. The QED₃ with a Maxwell term itself is very interesting and has been studied extensively in the past twenty years (for a review, see [24]). It also has direct applications in condensed matter physics. In the context of underdoped high temperature superconductors,

an effective QED₃ theory was derived to model the unusual physics after carefully considering the phase fluctuations [10, 11]. There is a Maxwell term in this kind of QED₃ theory. In the following, we will include the Maxwell term for gauge field in the Lagrangian of QED₃ and re-calculate the fermion damping rate.

In the presence of Maxwell term, the propagators for gauge field are

$$D_{00}(\Omega_m, \mathbf{q}) = \frac{1}{|\mathbf{q}|^2 + D_1(\Omega_m, \mathbf{q})}, \quad (66)$$

$$D_{ij}(\Omega_m, \mathbf{q}) = \left(\delta_{ij} - \frac{q_i q_j}{\mathbf{q}^2} \right) \frac{1}{|\mathbf{q}|^2 + \Omega_m^2 + D_\perp(\Omega_m, \mathbf{q})}. \quad (67)$$

After perturbative computations, we find the following transverse damping rate

$$\begin{aligned} \text{Im}\Sigma_T(\omega, \mathbf{k}) = & -\frac{4N}{\pi^2} \int d^2\mathbf{q} \frac{\sqrt{(\omega - |\mathbf{k} + \mathbf{q}|)^2 - |\mathbf{q}|^2}}{256 \left((\omega - |\mathbf{k} + \mathbf{q}|)^2 - |\mathbf{q}|^2 \right)^2 + N^2 \left((\omega - |\mathbf{k} + \mathbf{q}|)^2 - |\mathbf{q}|^2 \right)} \\ & \times \theta(\omega - |\mathbf{k} + \mathbf{q}| - |\mathbf{q}|). \end{aligned} \quad (68)$$

Since $\theta(\omega - |\mathbf{k} + \mathbf{q}| - |\mathbf{q}|) \equiv 0$ when $|\omega| \leq |\mathbf{k}|$, we still have $\text{Im}\Sigma_T(\omega \leq |\mathbf{k}|, \mathbf{k}) \equiv 0$. In the limit $|\mathbf{k}| = 0$, we get

$$\text{Im}\Sigma_T(\omega) = -\frac{1}{4\pi} \arctan\left(\frac{16\omega}{N}\right) + \frac{N}{64\pi\omega} - \frac{N}{64\pi\omega} \frac{N}{16\omega} \arctan\left(\frac{16\omega}{N}\right). \quad (69)$$

In the low-energy regime, $\omega \rightarrow 0$, it reduces to

$$\text{Im}\Sigma_T(\omega) = -\frac{8|\omega|}{3N\pi}, \quad (70)$$

which coincides with the standard behavior of marginal Fermi liquid. The general $\text{Im}\Sigma_T(\omega, \mathbf{k})$ can only be calculated by numerical methods, with results shown in Fig. 4.

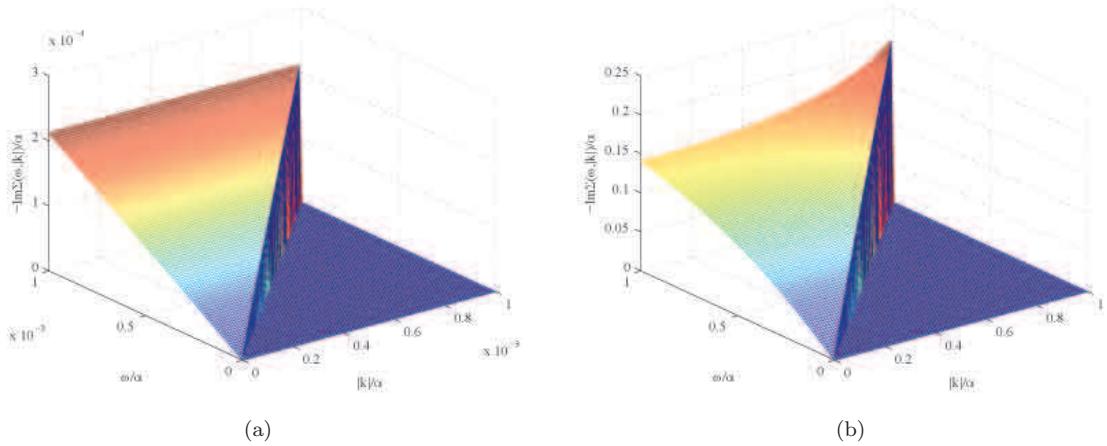


FIG. 4: Transverse contribution to damping rate for Dirac fermion in the presence of Maxwell term.

On the other hand, the longitudinal damping rate is found to be

$$\text{Im}\Sigma_L(\omega, \mathbf{k}) = -\frac{4N}{\pi^2} \int d^2\mathbf{q} \frac{\sqrt{(\omega - |\mathbf{k} + \mathbf{q}|)^2 - |\mathbf{q}|^2}}{256 \left((\omega - |\mathbf{k} + \mathbf{q}|)^2 - |\mathbf{q}|^2 \right) |\mathbf{q}|^2 + N^2 |\mathbf{q}|^2} \theta(\omega - |\mathbf{k} + \mathbf{q}| - |\mathbf{q}|) \quad (71)$$

It is easy to prove that $\text{Im}\Sigma_L(\omega \leq |\mathbf{k}|, \mathbf{k}) \equiv 0$. In the limit $|\mathbf{k}| = 0$,

$$\text{Im}\Sigma_L(\omega) = -\frac{8N}{\pi} \int_0^{\frac{\omega}{2}} d|\mathbf{q}| \frac{1}{|\mathbf{q}|} \left[\frac{\sqrt{\omega^2 - 2\omega|\mathbf{q}|}}{256(\omega^2 - 2\omega|\mathbf{q}|) + N^2} \right] \quad (72)$$

This is still divergent in the infrared region. The general $\text{Im}\Sigma_L(\omega, \mathbf{k})$ can be written as

$$\text{Im}\Sigma_L(\omega, \mathbf{k}) = -\frac{4N}{\pi^2} \int_0^{+\infty} d|\mathbf{q}| F_2(|\mathbf{q}|) \quad (73)$$

where

$$F_2(|\mathbf{q}|) = \int_0^{2\pi} d\varphi \frac{\sqrt{(\omega - |\mathbf{k} + \mathbf{q}|)^2 - |\mathbf{q}|^2}}{256 \left((\omega - |\mathbf{k} + \mathbf{q}|)^2 - |\mathbf{q}|^2 \right) |\mathbf{q}| + N^2 |\mathbf{q}|} \theta(\omega - |\mathbf{k} + \mathbf{q}| - |\mathbf{q}|) \quad (74)$$

When $\omega > |\mathbf{q}|$, we have

$$\lim_{|\mathbf{q}| \rightarrow 0} F_2(|\mathbf{q}|) = \frac{1}{|\mathbf{q}|} \left[\frac{2\pi(\omega - |\mathbf{k}|)}{256(\omega - |\mathbf{k}|)^2 + N^2} \right]. \quad (75)$$

Thus we can easily see that $\text{Im}\Sigma_L(\omega, \mathbf{k})$ is infrared divergent when $|\omega| > |\mathbf{k}|$.

Due to this divergence, the longitudinal fermion damping rate is still ill-defined at zero temperature $T = 0$, even in the presence of the Maxwell term.

At finite temperature, after tedious calculations we find that the Dirac fermion damping rate has the following features: If we calculate in the limit $|\mathbf{k}| = 0$ at $T \gg \omega$, then $\text{Im}\Sigma_L(\omega, T) \propto \sqrt{\omega T}$ and $\text{Im}\Sigma_T(\omega, T) \propto T/\omega$. If we work in the on-shell approximation $\omega = \omega_{\mathbf{k}} = |\mathbf{k}|$ at $T \gg \omega_{\mathbf{k}}$, then we find that $\text{Im}\Sigma_L(\omega_{\mathbf{k}}, T) \propto \sqrt{\omega_{\mathbf{k}} T}$ and that $\text{Im}\Sigma_T(\omega_{\mathbf{k}}, T)$ is divergent in the infrared region. In the zero-energy limit $\omega = 0$ with $T \gg |\mathbf{k}|$, we have $\text{Im}\Sigma_L(|\mathbf{k}|, T) \propto \sqrt{|\mathbf{k}| T}$ and $\text{Im}\Sigma_T(|\mathbf{k}|, T) \propto \sqrt{|\mathbf{k}| T}$.

All the above calculations are obtained in the Coulomb gauge, which separates the longitudinal and transverse components of the gauge field completely. The same calculations can be done similarly by choosing another gauge. After straightforward computation [25], we find that the fermion damping rate is still divergent in a general gauge when obtained at the perturbative level, no matter the Maxwell term of gauge field is present or not. This implies that the existence of divergence in perturbative expansion is a universal feature of QED₃, rather than just a gauge artifact.

The marginal Fermi liquid behavior of fermion damping rate was claimed previously by Franz and Tesanovic without providing computational details [10]. However, the detailed calculations show that the fermion damping rate is an ill-defined quantity at the perturbative level because divergence appears at both the $T = 0$ and $T \gg \omega$ limits.

We next turn to the self-consistent treatment of fermion damping rate in the presence of Maxwell term. To compare with the results presented above, we also choose to work in the Coulomb gauge. As in the last section, we include both the real and imaginary parts of the vacuum polarizations when writing the integral equation for the fermion damping rate

$$\begin{aligned} & \text{Im}\Sigma(\omega, T) \\ &= \frac{1}{2\pi^2} \int_0^{+\infty} d|\mathbf{q}| \int_{-\infty}^{+\infty} d\omega_1 \frac{\text{Im}\Sigma(\omega_1, T)}{(\omega_1 - |\mathbf{q}|)^2 + (\text{Im}\Sigma(\omega_1, T))^2} \\ & \times \left\{ -\text{Im} \left[\frac{1}{|\mathbf{q}|^2 + D_1(\omega_1 - \omega, |\mathbf{q}|, T)} \right] + \text{Im} \left[\frac{1}{|\mathbf{q}|^2 - (\omega_1 - \omega)^2 + D_{\perp}^{\text{ret}}(\omega_1 - \omega, |\mathbf{q}|, T)} \right] \right\} \\ & \times [n_B(\omega_1 - \omega) + n_F(\omega_1)], \end{aligned} \quad (76)$$

where the functions D_1 and D_{\perp} are given by equations (59-63) in the last section. The kinetic term of gauge field eliminated the divergence brought by the polarization functions. At the same time, the energy ω can no longer be scaled out due to the kinetic gauge term. Thus the fermion damping rate is not expected to display the marginal Fermi liquid behavior. After numerically solving these coupled integral equations, we find that the total fermion damping rate is

$$\text{Im}\Sigma(\omega) \propto \sqrt{\omega} \quad (77)$$

at zero temperature $T = 0$. The numerical results at $T = 0$ are presented in Fig. 5(a). The computation of fermion damping rate at finite temperature follows the same procedure as presented above. The numerical computations find the following fermion damping rate

$$\text{Im}\Sigma(T) \propto \sqrt{T} \quad (78)$$

in the limit $T \gg \omega$. The numerical results for this limit are presented in Fig. 5(b). As a summary, the damping rate of massless Dirac fermions is

$$\text{Im}\Sigma(\omega, T) \propto \max(\sqrt{\omega}, \sqrt{T}) \quad (79)$$

due to scattering by the U(1) gauge field in two spatial dimensions. This is certainly a non-Fermi liquid like behavior since the fermion damping rate would be $\text{Im}\Sigma(\omega, T) \propto \max(\omega^2, T^2)$ (or with higher powers) in a normal Fermi liquid.

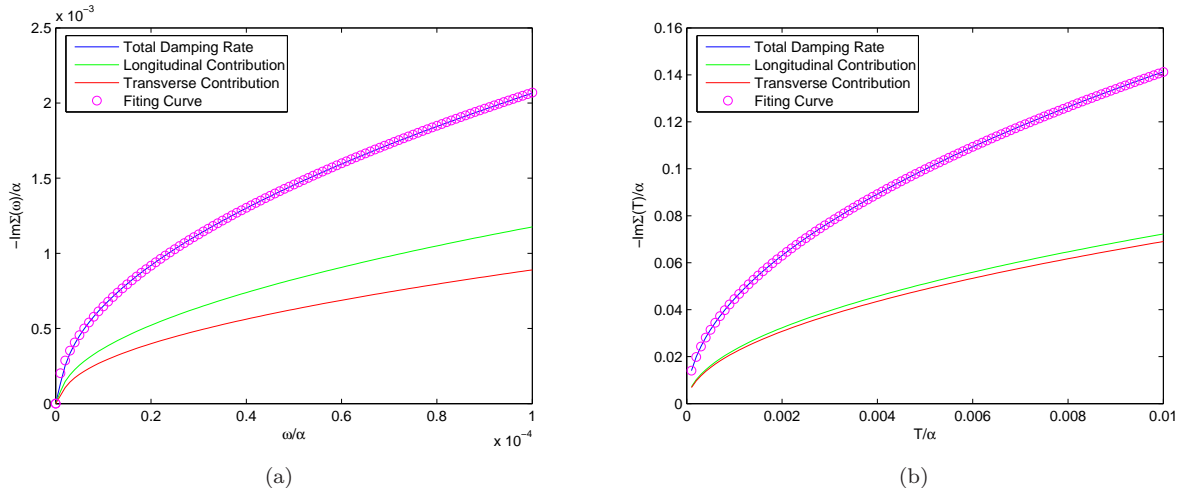


FIG. 5: (a) Damping rate of massless Dirac fermions at zero temperature $T = 0$. The fitting curve is $\text{Im}\Sigma(\omega)/\alpha = 0.2170(\omega/\alpha)^{0.5051}$. Here the energy scale α is defined as $\alpha = Ne^2/8$ (we have let $e = 1$ in this paper), which was first introduced in Ref. [17]. (b) Damping rate of massless Dirac fermions on Fermi level at finite temperature. The fitting curve is $\text{Im}\Sigma(T)/\alpha = 1.4220(T/\alpha)^{0.5014}$.

This treatment is essentially the analogy of Eliashberg theory of superconductors with strong electron-phonon interaction. The propagators of fermions and intermediate bosons are self-consistently coupled while the vertex corrections are simply ignored. In the standard Eliashberg theory of electron-phonon system, the vertex corrections can be safely neglected since they are suppressed by the small parameter m/M , where m and M are the electron mass and nuclei mass respectively. This is nothing but the Migdal theorem [26]. In the present problem, there is no similar mass scales since both Dirac fermions and gauge bosons are massless. However, we do have a small expansion parameter $1/N$. The vertex corrections are suppressed by the factor $1/N$. In the large N limit, our ignorance of the vertex corrections is valid (Polchinski used the same argument when studying the fermion self-energy due to gauge interaction, see his paper in Ref. [16]). The validity of our results for small N , such as $N = 2$, is not clear currently.

Unfortunately, at present we are unable to get a well-defined result for the momentum dependence of fermion damping rate from the same self-consistent treatment. The reason is that the coupled equations become very complicated after including the momentum dependence. The numerical results are much less reliable as those at zero momentum.

IV. MARGINAL FERMI LIQUID BEHAVIOR AND COULOMB INTERACTION

The marginal Fermi liquid was proposed by Varma *et al.* at 1989 to understand some of the highly unusual experimental facts in the normal state of high temperature superconductors on phenomenological grounds [21]. Since then, a lot of efforts have been devoted to deriving this phenomenological theory from certain microscopic models [1, 10, 15, 16]. Gauge theory has long been considered as one possible candidate [10, 15, 16]. Within the ordinary perturbative theory, the transverse damping rate of Dirac fermions has a linear dependence on energy at $T = 0$, which is the standard behavior of marginal Fermi liquid. However, this result can not be trusted because of the appearance of divergence in the longitudinal component at $T = 0$ and in the transverse component at finite temperature. More careful theoretical and numerical computations show that the Dirac fermions exhibit non-Fermi, rather than marginal Fermi, liquid behavior.

However, it is possible to find signature of marginal Fermi liquid within some models those are in form analogous to the U(1) gauge field theory. For instance, we now consider the long-range Coulomb interaction between Dirac fermions, which remains unscreened because of the vanishing density of states at the Fermi level. In some sense, the

Coulomb potential may be regarded as the temporal component of a U(1) gauge field. There is, however, a subtle difference, which leads to important consequence. The dynamically screened Coulomb interaction can be formally written as

$$D_C(i\Omega_m, \mathbf{q}) = \frac{1}{\frac{|\mathbf{q}|}{\lambda} + D_1(i\Omega_m, \mathbf{q})}, \quad (80)$$

where the $|\mathbf{q}|$ term comes from the bare, instantaneous Coulomb potential and the dimensionless parameter is defined as $\lambda = \frac{2\pi}{\epsilon_0}$ with ϵ_0 being the dielectric constant. At zero temperature, the damping rate of Dirac fermions has the form

$$\text{Im}\Sigma_C(\omega, \mathbf{k}) = -\frac{4N}{\pi^2} \int d^2\mathbf{q} \frac{\sqrt{(\omega - |\mathbf{k} + \mathbf{q}|)^2 - |\mathbf{q}|^2}}{256 \left((\omega - |\mathbf{k} + \mathbf{q}|)^2 - |\mathbf{q}|^2 \right)^{\frac{1}{\lambda^2}} + N^2 |\mathbf{q}|^2} \theta(\omega - |\mathbf{k} + \mathbf{q}| - |\mathbf{q}|) \quad (81)$$

Here, we also have $\text{Im}\Sigma_C(\omega \leq |\mathbf{k}|, \mathbf{k}) \equiv 0$. In the limit $|\mathbf{k}| = 0$, we get

$$\text{Im}\Sigma_C(\omega) = -\frac{8}{N\pi} C(\lambda) \omega \quad (82)$$

with

$$C(\lambda) = \int_0^1 dx \frac{x\sqrt{1-x}}{\left(\frac{32}{\lambda N}\right)^2 (1-x) + x^2}. \quad (83)$$

This damping rate is free of divergence and can be identified as the behavior of a marginal Fermi liquid. It agrees with the results obtained in the context of graphene [27–29].

The damping rate $\text{Im}\Sigma_C(\omega, \mathbf{k})$ at general energy-momentum is calculated by numerical methods, with results shown in Fig. 6.

We can also study the self-consistent equations of damping rate and polarization function. As in the case of gauge field, we consider only the zero-momentum limit. Now the equation for fermion damping rate is

$$\begin{aligned} \text{Im}\Sigma(\omega, T) = & -\frac{1}{2\pi^2} \int_0^{+\infty} d|\mathbf{q}| \int_{-\infty}^{+\infty} d\omega_1 \frac{\text{Im}\Sigma(\omega_1, T)}{(\omega_1 - |\mathbf{q}|)^2 + (\text{Im}\Sigma(\omega_1, T))^2} \text{Im} \left[\frac{1}{\frac{|\mathbf{q}|}{\lambda} + D_1(\omega_1 - \omega, \mathbf{q}, T)} \right] \\ & \times [n_B(\omega_1 - \omega) + n_F(\omega_1)], \end{aligned} \quad (84)$$

where $D_1(\varepsilon, \mathbf{q}, T)$ was given by (59) and (60). Due to the special form of the bare interaction function $|\mathbf{q}|^{-1}$, the energy ω can be completely scaled out in the whole set of integral equations. This is not possible for $|\mathbf{q}|^{-2}$, as discussed in Sec. 3. Moreover, such bare term ensures the absence of divergence. In the absence of any bare term, the re-scaling can still be performed, but the equation is divergent, as discussed in Sec. 2. Therefore, the Coulomb interaction is very special and turns out to be a good candidate for producing marginal Fermi liquid behavior.

With the help of scaling analysis, it is easy to show that $\text{Im}\Sigma(\omega) \propto \omega$ in the limit $\omega \gg T$ and $\text{Im}\Sigma(T) \propto T$ in the limit $\omega \ll T$, respectively. As before, we define $\frac{\text{Im}\Sigma(\omega)}{\omega} = A$ and $\frac{\text{Im}\Sigma(T)}{T} = B$, then their dependence on λ can be obtained by numerical computation. The results are presented in Fig. 7(a) and Fig. 7(b).

The linear dependence of fermion damping rate on energy/temperature may appear in other counterpart of the U(1) gauge interaction. It was discovered in the model of nodal quasiparticles coupled to critical fluctuation of some superconducting order parameter [23, 30]. This model, albeit having different physical contents, share one common feature with Coulomb interaction: the coupling of massless Dirac fermions to some singular boson mode. This is in form analogous to U(1) gauge interaction. However, the calculation of fermion damping rate caused by gauge interaction meets with divergences. According to our theoretical and numerical computations, it seems that the kinetic term for gauge field has to be explicitly included in order to get meaningful results.

Aji and Varma constructed an interesting dissipative quantum 2D XY model and showed that it produces marginal Fermi liquid behaviors [31]. Recently, behavior of marginal Fermi liquid type was also found in a d -dimensional field theory with the help of AdS/CFT correspondence [32].

We also note that a linear-in- T quasiparticle damping rate was already pointed out in two early papers [14, 15]. In both of these works, the linear-in- T behavior is attributed to the scattering of quasiparticles by an emergent U(1) gauge field. However, such damping rate is that of spinless holons, rather than fermions. Therefore, the interacting system considered in these papers, though very interesting, can not be identified as a marginal "Fermi" liquid.

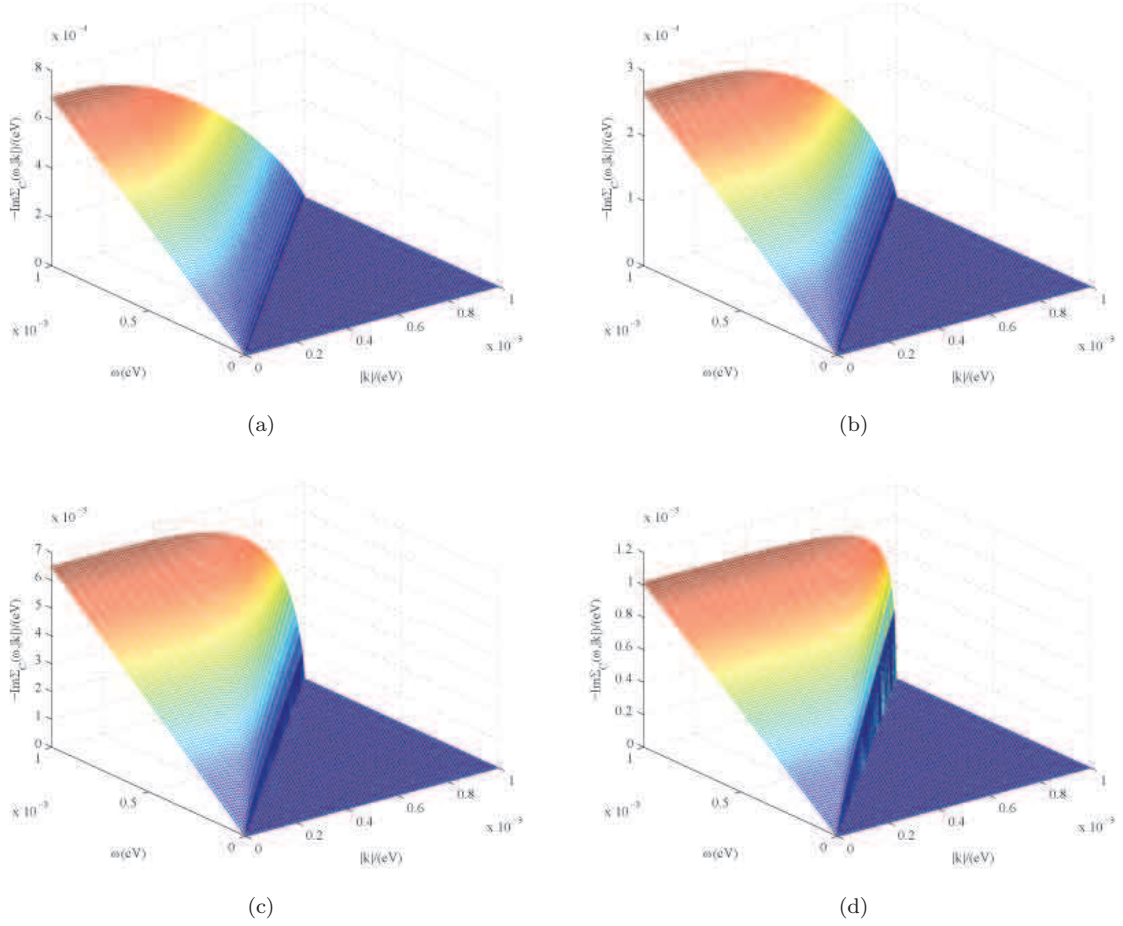


FIG. 6: Damping rate for Dirac fermion due to the Coulomb interaction. λ^{-1} are 0.001, 0.01, 0.1, and 1 in (a), (b), (c), (d) respectively.

V. SUMMARY AND DISCUSSION

In summary, we present a detailed calculation of the damping rate of massless Dirac fermions due to U(1) gauge field in QED₃. When the theory contains no Maxwell term for gauge field, the fermion damping rate is found to diverge at both zero and finite temperatures within perturbation theory. There is still divergence in the self-consistent equations for fermion damping rate and gauge boson propagator. Once the Maxwell term for gauge field is included into the self-consistent equations, the fermion damping rate is well-defined and display non-Fermi liquid behaviors at both zero and finite temperatures.

From the first sight, the existence of divergence in fermion damping rate in QED₃ theory without Maxwell term might restrict its validity in understanding high temperature superconductors. However, in reality this is not as severe as it looks. Such effective theory applies to both the half-filling state and the underdoped region of high temperature superconductors. In the half-filling state, the massless Dirac fermions undergo a pairing instability towards the chiral symmetry breaking phase. Now the fermionic excitations are suppressed by the dynamically generated mass gap and thus it is usually not necessary to study the damping rate. In the underdoped region, the gauge field couples not only to massless Dirac fermions, but also to an additional boson field which describes the motion of charged holons. The interaction between gauge field and holons contributes a vacuum polarization function to the gauge boson propagator, which might be able to eliminate the divergence appearing in the fermion damping rate. To address this issue, it is essential to carefully study the whole interacting system, especially the holon-gauge coupling.

At present, we are unable to eliminate the divergence appearing in the momentum dependence of Dirac fermion damping rate. Such divergence might also be cured by the self-consistent (Eliashberg) treatment, but the coupled equations become much more complicated than the zero-momentum limit and are hence hard to be solved numerically. We will study this problem further, either by improving numerical methods or by developing novel theoretical

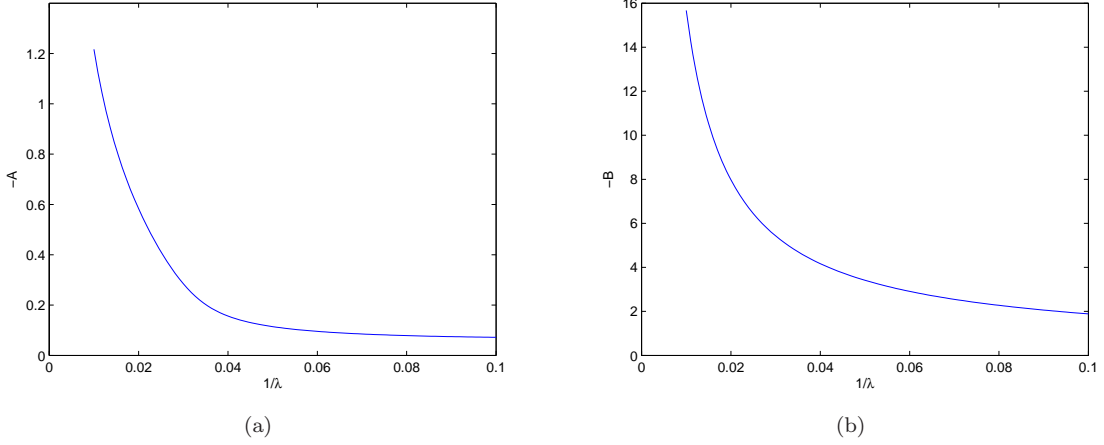


FIG. 7: (a) The relation between A and $1/\lambda$. (b) The relation between B and $1/\lambda$.

approaches.

When the U(1) gauge theory is used to describe the anomalous properties of high temperature superconductors, the Dirac fermions are usually not the physical electrons. They might be fermionic spinons [3, 5–7], fermionic holons [9], or topological fermions [10, 11], depending on the physical motivations and the procedures of deriving the effective field theory. Therefore, the damping rate studied in this work could be directly compared with experiments only after including the additional degrees of freedom. However, since the U(1) gauge interaction of massless Dirac fermions appears naturally in a number of correlated electron systems, we believe it is interesting to carefully study the damping rate and other physical quantities of the “unphysical” Dirac fermions.

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